IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

New conserved quantities derived from symmetry for stochastic dynamical systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1994 J. Phys. A: Math. Gen. 27 L777

(http://iopscience.iop.org/0305-4470/27/20/004)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 01/06/2010 at 22:50

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 27 (1994) L777–L782. Printed in the UK

LETTER TO THE EDITOR

New conserved quantities derived from symmetry for stochastic dynamical systems

Tetsuya Misawa

Department of Mathematics, College of General Education Nagoya City University, Mizuho-ku, Nagoya 467, Japan

Received 3 August 1994

Abstract. Recently, the author has proposed an elementary theory of conserved quantities and symmetry for stochastic dynamical systems described by stochastic differential equations of Stratonovich type. Within the framework, a new method for deriving conserved quantities from symmetry is developed in a similar manner to that of Hojman's work which gives a new conservation law constructed without using either Lagrangians or Hamiltonians for deterministic dynamical systems. Some examples of the conserved quantities obtained through the new method are given.

The theory of conserved quantities (the first integrals) and symmetry (invariant under a transformation) for dynamical systems must be one of the most important subjects in physics. Hence, it is natural to formulate these notions for stochastic dynamical systems described by stochastic differential equations. Indeed, there has been increasing interest in stochastic models for nonlinear integrable systems which have essentially some conserved quantities (e.g. Lotka–Volterra system) (Itoh 1993, Nakamura 1994). In consideration of these facts, the author has recently proposed an elementary theory of conserved quantities and symmetry for the stochastic dynamical systems (Misawa 1994).

On the other hand, in the theory of deterministic dynamical systems, it is widely known that most of the conservation laws are derived from the Lagrangian or the Hamiltonian structures of the equation describing the system. Hence, if dynamical systems do not have such structures, it may be difficult to find out conserved quantities for the systems. However, Hojman has proposed a new method for deriving a conserved quantity from the infiniesimal symmetry transformation of the dynamical equation only (Hojman 1992, Mimura and Nono 1994). The main theorem he verified is outlined as follows: Let us consider the dynamical system of n second-order differential equations

$$\ddot{q}^i - F^i(\dot{q}, q, t) = 0 \qquad (i = 1, 2, \ldots n)$$

with the condition $\sum_{i=1}^{n} \partial F^i/\partial \dot{q}^i = 0$, where $q = (q^i(t))_{i=1}^n$, $\dot{q} = (\mathrm{d}q^i/\mathrm{d}t)_{i=1}^n$ and $\ddot{q} = (\ddot{q}^i)_{i=1}^n = (\mathrm{d}^2 q^i/\mathrm{d}t^2)_{i=1}^n$. Suppose that this system is invariant under the infinitesimal symmetry equation of the transformation $\tilde{q}^i = q^i + af^i(\dot{q}, q, t)$ $(i = 1, 2, \ldots n)$ up to a^2 terms, where a is an infinitesimal parameter. Then, the function $f = (f^i)_{i=1}^n$ yields the following conserved quantity for the system:

$$I = \sum_{i=1}^{n} \frac{\partial f^i}{\partial q^i} + \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}^i} \left(\frac{Df^i}{Dt} \right)$$

where D/Dt is the following operator:

$$\frac{D}{Dt}=\sum_{i=1}^{n}F^{i}\frac{\partial}{\partial\dot{q}^{i}}+\sum_{i=1}^{n}\dot{q}^{i}\frac{\partial}{\partial q^{i}}+\frac{\partial}{\partial t}.$$

Thus, a conserved quantity is obtained through symmetry for the dynamical system. Here we remark that the quantity given in the left-hand side of the condition for 'an external force' $\boldsymbol{F}=(F^{i})_{i=1}^{n}$ and the conserved quantity mentioned above are a sort of 'divergence'.

In the stochastic systems we investigate, the Lagrangian or the Hamiltonian structures often fail to exist. Therefore, it must be important and relevant to formulate a similar theorem to that of Hojman for our stochastic dynamical systems; and this is the main purpose of the present article.

In what follows, we consider stochastic dynamical systems described by the following n-dimensional vector valued stochastic differential equations of Stratonovich type (Ikeda and Watanabe 1985):

$$\mathrm{d}x_t=b(x_t,t)\,\mathrm{d}t+\sum_{r=1}^{m}g_r(x_t,t)\circ\mathrm{d}w_t^r \qquad x_{t_0}=c,\ t\in[t_0,T] \tag{1}$$

where $w_t=(w_t^r)_{r=1}^{m}$ is an m-dimensional standard Wiener process, c is a constant n-vector, and $b=(b^i)_{i=1}^{n}$ and $g_r=(g_r^i)_{i=1}^{n}$ are n-dimensional smooth functions, respectively (for convenience, differentiability of any function in our article is assumed to be of sufficiently high order). Now, to make this article self-contained, we repeat here the basic notions and results for conserved quantities and symmetry for the system (1) (Misawa 1994). Let I be a smooth function on $R^n\times R^1$, and ∂_t, X_0 and X_r $(r=1,2,\ldots m)$ are differential operators defined by

$$\partial_t=\frac{\partial}{\partial t} \qquad X_0=\sum_{i=1}^{n}b^i\frac{\partial}{\partial x_i} \qquad X_r=\sum_{i=1}^{n}g_r^i\frac{\partial}{\partial x_i}. \tag{2}$$

Then, we call the function I a *conserved quantity for a stochastic dynamical system* (1) if it satisfies

$$(\partial_t+X_0)I(x,t)=0 \qquad X_rI(x,t)=0 \qquad (r=1,2,\ldots m). \tag{3}$$

Indeed, using the change of variables formula for stochastic differential equations of Stratonovich type (Ikeda and Watanabe 1985) and (3), we obtain the following equation

$$\mathrm{d}I(x_t,t)=(\partial_t+X_0)I(x_t,t)\,\mathrm{d}t+\sum_{r=1}^{m}X_rI(x_t,t)\circ\mathrm{d}w_t^r=0 \tag{4}$$

where x_t is a diffusion process governed by (1). This indicates that '$I(x_t,t)=$ constant' holds on the diffusion process x_t; hence, we may regard I satisfying (3) as a conserved quantity for (1).

In a similar fashion, we define symmetry for stochastic dynamical systems (1). Let $y=\phi(x,t)$ be a transformation from $R^n\times R^1$ to R^n. We call ϕ a *symmetry transformation for a stochastic dynamical system* (1) if the function satisfies

$$b(\phi(x,t),t)=(\partial_t+X_0)\phi(x,t) \qquad g_r(\phi(x,t),t)=X_r\phi(x,t) \qquad (r=1,2,\ldots m). \tag{5}$$

Because, by the change of variables formula and (5), we have the following stochastic differential equation describing the process $y_t = \phi(x_t, t)$

$$\begin{aligned} \mathrm{d}y_t &= (\partial_t + X_0)\phi(x_t, t)\,\mathrm{d}t + \sum_{r=1}^{m} X_r\phi(x_t, t) \circ \mathrm{d}w_t^r \\ &= b(y_t, t)\,\mathrm{d}t + \sum_{r=1}^{m} g_r(y_t, t) \circ \mathrm{d}w_t^r \end{aligned} \tag{6}$$

where x_t is a diffusion process governed by (1). This means that a stochastic dynamical system described by (1) is invariant under the transformation satisfying (5); hence, such a transformation may be called 'symmetry'. On the basis of this definition, we can further formulate the notion of symmetry operators. Consider the differential operator $Y = \sum_{i=1}^{n} f^i(x, t)\partial/\partial x^i$, where $f = (f^i)_{i=1}^{n}$ is an R^n-valued smooth function. Moreover, let $y = \phi(x, t; a)$ be a local one-parameter transformation generated by Y (Eisenhart 1961), where a is a parameter on $T = (-a_0, a_0)$ and $\phi(x, t; 0) = x$. Here we assume that $\phi(x, t; a)$, $b(\phi(x, t; a), t)$ and $g_r(\phi(x, t; a), t)$ $(r = 1, 2, \ldots m)$ are analytic with respect to the parameter a on T. Then, using the definition of one-parameter transformation and the expansions of ϕ, $b(\phi, t)$ and $g_r(\phi, t)$ with respect to a, we can verify the following theorem:

Theorem 0. The one-parameter transformation generated by a differential operator Y is a symmetry transformation of a stochastic dynamical system (1) if, and only if, the operator Y satisfies

$$[\partial_t + X_0, Y] = 0 \qquad [X_r, Y] = 0 \qquad (r = 1, 2, \ldots m) \tag{7}$$

where $[\cdot, \cdot]$ denotes Lie bracket and ∂_t stands for the operator $\partial/\partial t$.

On account of this theorem, we call Y satisfying (7) a *symmetry operator* for (1). In particular, if the one-parameter transformation $\phi(x, t; a)$ is given as an infinitesimal transformation $\phi(x, t; a) = x + af(x, t)$ (a the infinitesimal parameter), theorem 0 reduces to the following:

Theorem 0′. A stochastic dynamical system (1) is invariant up to a-terms under the infinitesimal transformation derived from a differential operator Y if, and only if, Y is a symmetry operator for (1).

Thereby, such an infinitesimal transformation is called 'infinitesimal symmetry'. In the author's paper (Misawa 1994), several examples with respect to conserved quantities and symmetry for stochastic nonlinear systems involving Lotka–Volterra systems are given.

Now, we are to formulate a similar theorem to that of Hojman within the framework of our stochastic systems mentioned above. In consideration of the remark in the introductory part of the present article, we may expect that such a theorem is given as follows:

Theorem 1. For given stochastic dynamical system (1), assume that b and g_r $(r = 1, 2, \ldots m)$ satisfy

$$\operatorname{div} b = 0 \qquad \operatorname{div} g_r = 0 \qquad (r = 1, 2, \ldots m). \tag{8}$$

Then, the function $f = (f^i)_{i=1}^{n}$ in a symmetry operator $Y = \sum_{i=1}^{n} f^i(x, t)\partial/\partial x^i$ for the system (1) (i.e. the function f satisfying (7)) yields the following conserved quantity for (1):

$$I = \operatorname{div} f. \tag{9}$$

In fact, more generally, we can prove the following theorem:

Theorem 1′. For given stochastic dynamical system (1), assume that there exists a function $\varphi = \varphi(x, t)$ satisfying

$$\operatorname{div} \boldsymbol{b} + (\partial_t + X_0)\varphi = 0 \qquad \operatorname{div} \boldsymbol{g}_r + X_r\varphi = 0 \qquad (r = 1, 2, \ldots m) \quad (10)$$

where X_0 and X_r are the operators given by (2). Then, the function $\boldsymbol{f} = (f^i)_{i=1}^n$ in a symmetry operator $Y = \sum_{i=1}^n f^i(x, t)\partial/\partial x^i$ for (1) yields the following conserved quantity for the system:

$$I = \operatorname{div} \boldsymbol{f} + Y\varphi. \quad (11)$$

Theorem 1 is just a particular case of theorem 1′. Indeed, if we set φ as 0, (10) reduces to (8). Then, of course, (11) just coincides with (9). Hence, we will only prove theorem 1′.

Proof of Theorem 1′. To show theorem 1′ we have to examine that function I given by (11) satisfies (3); this is a matter of calculation. First, inserting (11) into the left-hand side of each equation in (3), we have

$$(\partial_t + X_0)I = (\partial_t + X_0)(\operatorname{div} \boldsymbol{f}) + (\partial_t + X_0)Y\varphi \quad (12a)$$

$$X_r I = X_r(\operatorname{div} \boldsymbol{f}) + X_r Y\varphi \qquad (r = 1, 2, \ldots m). \quad (12b)$$

Using (7) and (10), we can rewrite the second terms of the right-hand sides of (12*a*) and (12*b*) as

$$(\partial_t + X_0)Y\varphi = Y(\partial_t + X_0)\varphi = -Y(\operatorname{div} \boldsymbol{b}) \quad (13a)$$

$$X_r Y\varphi = Y X_r\varphi = -Y(\operatorname{div} \boldsymbol{g}_r) \quad (13b)$$

respectively. On the other hand, by simple computation, we see that the first terms of them are expressed as

$$\begin{aligned}(\partial_t + X_0)(\operatorname{div} \boldsymbol{f}) &= \sum_{i=1}^n (\partial_t + X_0)\partial_i f^i \\ &= \sum_{i=1}^n \partial_i\{(\partial_t + X_0) f^i\} - \sum_{i,j=1}^n \partial_i b^j \partial_j f^i \end{aligned} \quad (14a)$$

$$\begin{aligned}X_r(\operatorname{div} \boldsymbol{f}) &= \sum_{i=1}^n X_r \partial_i f^i \\ &= \sum_{i=1}^n \partial_i (X_r f^i) - \sum_{i,j=1}^n \partial_i g_r^j \partial_j f^i \end{aligned} \quad (14b)$$

respectively, where $\partial_i = \partial/\partial x^i$ $(i = 1, 2, \ldots n)$. Here, we remark that the following equations are derived from (7):

$$(\partial_t + X_0) f^i = Y b^i = \sum_{j=1}^n f^j \partial_j b^i$$

$$X_r f^i = Y g_r^i = \sum_{j=1}^n f^j \partial_j g_r^i.$$

The substitutions of these equations into (14*a*) and (14*b*) yield

$$(\partial_t + X_0)(\mathrm{div}\, f) = \sum_{i,j=1}^{n} \partial_i f^j \partial_j b^i + \sum_{i,j=1}^{n} f^j \partial_i \partial_j b^i - \sum_{i,j=1}^{n} \partial_i b^j \partial_j f^i$$
$$= \sum_{i,j=1}^{n} f^j \partial_j \partial_i b^i = Y(\mathrm{div}\, b) \qquad (15a)$$

$$X_r(\mathrm{div}\, f) = \sum_{i,j=1}^{n} \partial_i f^j \partial_j g_r^i + \sum_{i,j=1}^{n} f^j \partial_i \partial_j g_r^i - \sum_{i,j=1}^{n} \partial_i g_r^j \partial_j f^i$$
$$= \sum_{i,j=1}^{n} f^j \partial_j \partial_i g_r^i = Y(\mathrm{div}\, g_r). \qquad (15b)$$

The equations (13*a*), (13*b*) and (15*a*), (15*b*) imply that (12*a*) and (12*b*) turn out to be

$$(\partial_t + X_0)I = 0 \qquad \text{and} \qquad X_r I = 0 \qquad (r = 1, 2, \ldots m)$$

respectively, and thereby completing the proof of theorem 1′.

At the end of our article, we will give two examples of conserved quantities obtainable by applying theorem 1 or theorem 1′ to the stochastic dynamical systems (1).

Example 1. Consider the following 3-dimensional stochastic linear dynamical system:

$$\mathrm{d}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} (x_t^3 - x_t^2) \\ (x_t^1 - x_t^3) \\ (x_t^2 - x_t^1) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} (x_t^3 - x_t^2) \\ (x_t^1 - x_t^3) \\ (x_t^2 - x_t^1) \end{pmatrix} \circ \mathrm{d}w_t. \qquad (16)$$

Then, this system satisfies the condition (8) in theorem 1; indeed, we can easily check that $\mathrm{div}\, b = 0$ and $\mathrm{div}\, g_1 = 0$ hold. We further find out that $Y = \sum_{i=1}^{3} f^i(x,t)\partial_i = \{(x^1)^2 + (x^2)^2 + (x^3)^2\}(\partial_1 + \partial_2 + \partial_3)$ becomes a symmetry operator for this system, since the operator satisfies (7) for (16) (note that ∂_i stands for $\partial/\partial x^i$). Hence, we can apply theorem 1 to this system; it proves that the following scalar function is a conserved quantity for the system:

$$I = \sum_{i=1}^{3} \partial_i f^i = \sum_{i=1}^{3} \partial_i \{(x^1)^2 + (x^2)^2 + (x^3)^2\}$$
$$= 2(x^1 + x^2 + x^3). \qquad (17)$$

We remark that this result is directly examined through (3) and (17).

Example 2. Next we work with the following 4-dimensional stochastic nonlinear dynamical system:

$$\mathrm{d}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ x_t^4 \end{pmatrix} = \begin{pmatrix} (x_t^1)^2 \\ x_t^1 \cdot x_t^2 \\ x_t^1 \cdot x_t^3 \\ x_t^1 \cdot x_t^4 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} x_t^1 \cdot x_t^2 \\ (x_t^2)^2 \\ x_t^2 \cdot x_t^3 \\ x_t^2 \cdot x_t^4 \end{pmatrix} \circ \mathrm{d}w_t. \qquad (18)$$

Then, we can choose the function $\varphi(x,t)$ satisfying (10) in theorem 1′ as

$$\varphi(x,t) = -5\log x^1 + \frac{x^3}{x^4}.$$

Moreover, it is easily verified that $Y = \sum_{i=1}^{4} f^i(x,t)\partial_i = (x^1 \cdot x^4/x^2)\partial_4$ is a symmetry operator for this stochastic system (18). Hence, we may apply theorem 1′ to the system, and thereby, we get the following conserved quantity.

$$\begin{aligned} I &= \sum_{i=1}^{4} \partial_i f^i + Y\varphi \\ &= \partial_4\left(\frac{x^1 \cdot x^4}{x^2}\right) + \left(\frac{x^1 \cdot x^4}{x^2}\right)\partial_4\left(-5\log x^1 + \frac{x^3}{x^4}\right) \\ &= \frac{(x^1 \cdot x^4 - x^1 \cdot x^3)}{x^2 \cdot x^4}. \end{aligned} \tag{19}$$

Note that this result is also directly confirmed through (3) and (19).

This work was partially supported by Grant-in-Aid for Scientific Research No 06740172 from the Ministry of Education, Science and Culture.

References

Eisenhart L P 1961 *Continuous Groups of Transformations* (New York: Dover)
Hojman S 1992 *J. Phys. A: Math. Gen.* L291–5
Ikeda N and Watanabe S 1985 *Stochastic Differential Equations and Diffusion Processes* 2nd edn (Amsterdam/Tokyo: North-Holland/Kodansha)
Itoh Y 1993 *J. Phys. Soc. Japan* **62** 1826–8
Mimura F and Nono T 1994 A new conservation law for a system of second-order differential equations *Bull. Kyushu Inst. Techn. Math. Natur. Sci.* **41** in press
Misawa T 1994 Conserved quantities and symmetry for stochastic dynamical systems (to be submitted)
Nakamura Y 1994 *J. Phys. Soc. Japan* **63** 827–9